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PROGRESS REPORT.

Mr. S. Krishnaswami, B.A., Mathematics Assistant, Mahajana High School, Erode, has been elected a member at concessional rate.

2. The "*Circolo Matematico di Palermo*" of Palermo, (Italy) having honoured our Society with exchange relations, the publications of that Society, viz., (i) Rendiconti. (ii) Supplemento, (iii) Annuario, and (iv) Indici, will be available for issue at the Library as will be notified from time to time by our Librarian.

3. The following have been received as presents from their publishers:

1. *Higher Algebra* for Colleges and Secondary Schools—by C. Davison, Sc.D., Camb. Univy. Press, 1912, 6*s*.
2. *Calcutta University Calendar*, Parts II & III for 1912.

POONA,
30th September, 1912.

D. D. KAPADIA,
Hony. Joint Secretary.

Approximations in Curve Tracing.

By M. T. Naraniengar.

The *form* of a curve at an infinite distance, or in the neighbourhood of the origin when the curve passes through it, is in all cases determined by approximations. That is, certain terms of the equation of the curve are rejected, and simpler curves found thereby, which nearly coincide with the original curve at infinity or near the origin. Newton's parallelogram and De Gua's triangle are well-known aids in determining the relative importance of the terms of an equation near specified points. By their means simple parabolic curves can always be found which are *first* approximations to the shape of the curve. The object of the present note is to examine the *second* and *further* approximations; and also to apply the known methods to some cases of *freedom equations.*

[References: Frost's *Curve Tracing*; Edward's *Treatise on the Differential Calculus*; and Lodge's *Differential Calculus for Beginners.*]

At Infinity:

1. Take the general equation of a curve

$$x^n\phi(\mu)+x^{n-1}\psi(\mu)+x^{n-2}\chi(\mu)+\ldots=0, \quad \ldots \quad \ldots \quad (1)$$

where μ is written for y/x, and ϕ, ψ, χ, ... denote functions of the nth, $(n-1)^{th}$, $(n-2)^{th}$, ... degrees.

If $\phi(\mu)=0$, has its roots all different, the n linear asymptotes and the corresponding approximations to the infinite branches of the curve are readily found by Sterling's method of expansion.

If $\phi(\mu)=0$, has two equal roots μ_1, there will be two parallel linear asymptotes, only provided $\psi(\mu_1)=0$, and the approximations to the corresponding infinite branches can be determined by the same method.

If, however, $\phi(\mu)=0$, has two equal roots μ_1 without $\psi(\mu_1)$ being equal to zero, the usual method of expansion fails and we must have recourse to the form

$$y=\mu_1 x+Ax^{\frac{1}{2}}+B+Cx^{-\frac{1}{2}}+ \quad \ldots \quad \ldots \quad (2).$$

[Edwards: § 262.]

Following this method Edwards obtains the parabolic asymptote

$$(y-\mu_1 x-B)^2=A^2x, \quad \ldots \quad \ldots \quad (3).$$

and observes that the side of the parabola on which the curve lies is indicated by the sign of C.

Now, there is a *closer* approximation to (2) than (3), *viz*:

$$(y-\mu_1 x+B)^2=A^2x+2\,AC, \quad \ldots \quad \ldots \quad (4).$$

which must therefore be regarded as the *true* parabolic asymptote of the curve, and not (3). The side of this parabola on which the curve lies is determined by writing down the expansion of y from (4), and comparing with (2). Thus, from (4) we have

$$y=\mu_1 x+Ax^{\frac{1}{2}}+B+Cx^{-\frac{1}{2}}-\tfrac{1}{2}\, C^2/A.x^{-\frac{3}{2}} \quad \ldots,$$

shewing that the sign of D determines the relative positions of the two curves.

Examples :

(i) $(y-x)^2\,(y+x)=2\,ax^2$.

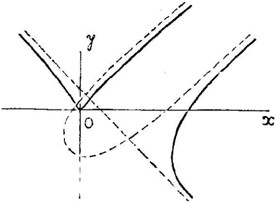
Put $y=x+Ax^{\frac{1}{2}}+B+Cx^{-\frac{1}{2}}+Dx^{-1}+\ldots\ldots$ and we get, by comparing co-efficients,

$$2\,A^2=2\,a,\ 4\,AB+A^3=0,$$
$$2\,(2\,AC+B^2)+2\,A^2B+A^2B=0,$$
$$2\,(2\,AD+2\,BC)+A\,(2\,AC+B^2)+B\,(2AB)+CA^2=0,$$
$$\therefore\ A=a^{\frac{1}{2}},\ B=-\frac{a}{4},\ C=\frac{5}{32}\,a^{\frac{3}{2}},\ D=-\frac{3}{32}\,a^2.$$

Hence the true parabolic asymptote is

$$\left(y-x+\frac{a}{4}\right)^2=ax+\frac{5}{16}\,a^2,$$

the ordinate of which is greater than that of the curve. Thus the upper branch of the curve approaches the parabola from the inside and the lower branch from the outside; and a tracing of the curve is as under. (Compare : Edwards, § 263, Ex.)



(ii) $x\,(y-x)^2=ay^2$.

Here we find $y-x=\sqrt{ax}+a+\sqrt{a^3/x}+a^2/x+\ldots\ldots$, and the true parabolic asymptote is

$$(y-x)^2-2\,ay+ax-a^2=0,$$

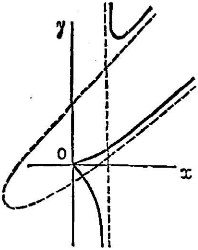
and not

$$(y-x)^2-2\,ay+ax+a^2=0,$$

as given in Frost, and the value of D *viz.*, a^2 shews that the curve approaches the parabola from the out-side at the upper branch, and from

the inside at the lower branch. [Compare: Frost, § 131, (1).]

A tracing of the curve is added:



(iii) $y^2(x^2-y^2)-2ax(x+2y)(x-y)-a^2(x+y)^2+2a^4=0.$

Here $$y=\sqrt{2ax}+a-\tfrac{1}{2}a\sqrt{2a/x} \quad \ldots \quad \ldots,$$

and therefore the parabolic asymptote is

$$(y-a)^2=2a(x-a),$$

agreeing with the result given in Math. Tripos, 1882.

(iv) $$y(y-x)^2(y+2x)=9cx^3$$

In this case

$$y=x+\sqrt{3cx}-2c+\tfrac{17}{2}8\sqrt{c/3x},$$

so that the parabolic asymptote of closest contact is

$$(y-x+2c)^2=2cx+17c^2.$$

But Frost gives

$$(y-x+2c)^2=3c(x+\tfrac{3}{2}c),$$

which must be incorrect. [Frost, §. 131, (2)].

2. It may be noted that in the general case considered in the previous article, the values of A,B,C are determined by the following equations:

$$\tfrac{1}{2}A^2\phi_2+\psi=0,\ AB\phi_2+\tfrac{1}{6}A^3\phi_3+A\psi_1=0,$$

and $$\tfrac{1}{2}(2AC+B^2)\phi_2+\tfrac{1}{2}A^2B\phi_3+\tfrac{1}{24}A^4\phi_4+B\psi_1+\tfrac{1}{2}A^2\psi_2+\chi=0,$$

where the suffixes denote differentiation with respect to μ_1.

3. The method of successive approximations may also be used, with advantage, in a majority of cases. Care, however, is necessary to see that the rules for approximation are strictly followed. In the examples considered in § 1 the approximations must be carried to the *third* order to get the true asymptotic curves.

The method of division explained in Lodge's book seems, however, the shortest approach to the *rationalized* form of the asymptote. As an illustration, let us take the curve

$$(y-x)^2(y+x)=2ax^2.$$

Write z for $y-x$ and arrange the equation in terms of $x+z$. We have

$$z^2(2x+z)=2ax^2.$$

$$\therefore \qquad z^2=\frac{2ax^2}{2x+z}.$$

Now, performing the division actually and approximating to the value of z in the third step, we find that $z^2=ax-\frac{1}{2}az+\frac{1}{4}a^2$, which agrees with what we found in §. 1,(i) above.

$$\begin{array}{l} 2x+z)\,2ax^2\;(ax-\frac{1}{2}az+\frac{1}{4}a^2 \\ \quad\;\; \underline{2ax^2+axz} \\ \quad\;\; -axz \\ \quad\;\; \underline{-axz-\frac{1}{2}az^2} \\ \quad\;\; \frac{1}{2}az^2=\frac{1}{2}a^2x \\ \qquad\quad \underline{\frac{1}{2}a^2x+\frac{1}{4}a^2z} \\ \qquad\qquad -\frac{1}{4}a^2z \end{array}$$

4. If, in the general case, $\phi\,(\mu)=0$ has three equal roots μ_1 and $\psi(\mu_1)\neq 0$, the asymptote is of the form

$$y=\mu_1x+Ax^{\frac{2}{3}}+Bx^{\frac{1}{3}}+C+Dx^{-\frac{1}{3}}+ \qquad \cdots \qquad \cdots \quad (5)$$

where A,B,C...must satisfy the equations

$$\frac{A^3\phi_3}{\underline{|3}}+\psi=0,\quad \frac{3A^2B\phi_3}{\underline{|3}}+\frac{A^4\phi_4}{\underline{|4}}+A\psi_1=0,$$

$$\frac{3\,(AB^2+A^2C)\,\phi_3}{\underline{|3}}+\frac{4A^3B\phi_4}{\underline{|4}}+\frac{A^5\phi_5}{\underline{|5}}+B\psi_1+A^2\psi_2=0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The rational integral curve of closest contact corresponding to (5) will not, in general, be simple.

Examples:

(i) $x^2\,(y-x)^3-ay^4=0.$ [Frost, § 143, (2).]

The method of successive approximations gives

$$y=x+a^{\frac{1}{3}}\,x^{\frac{2}{3}}+\tfrac{4}{3}\,a^{\frac{2}{3}}\,x^{\frac{1}{3}}+2a\,;$$

the last term is incorrectly given in Frost. At first sight, it may appear that the *rational* curve corresponding to this approximation is

$$(y-x-2\,a)^3=ax^2+\frac{64}{27}\,a^2x+4\,ax\,(y-x-2\,a).$$

But a careful examination of the method of approximation will show that the result is wrong.

Further, it will be found that Prof. Lodge's method of division *fails* in this case.

(ii) Consider the curve

$$x\,(y-x)^3=a\,y^2\,(3x-y).$$

Here, the method of division may be employed successfully. Thus, if we put $y-x=z$,

$$xz^3 = a\,(x+z)^2\,(2x-z)$$

$$\therefore\quad z^3 = a\,(2\,x^3+3\,x^2z-z^3)/x$$

$$= a\,(2\,x^2+3\,xz-z^3/x)$$

$$= a\,(2x^2+3xz-2ax-3az),$$

substituting for z^3 its approximate value $(2ax^2+3axz)$.

Hence the cubic asymptote is

$$(y-x)^3=3axy-ay^2-3a^2y+a^2x.$$

The method of approximation gives

$$z/x=\lambda^{\frac{1}{3}}+\tfrac{1}{2}\,\lambda^{\frac{2}{3}}-\frac{5}{24}\lambda^{\frac{4}{3}}\;\ldots\ldots\;,$$

where λ stands for $(2a/x)$, and the deduction of the approximate rationalized result as obtained above is tedious.

At the origin:

5. Let us next discuss the osculating curves near the origin, when the curve passes though the origin.

If the tangents at the origin be all different the usual method of approximation, or the expansion of y by Maclaurin's Theorem will do. If, however, two or more tangents coincide special forms for y have to be taken.

Thus, when two tangents coincide we should write

$$y=px+qx^{\frac{3}{2}}+rx^2+sx^{\frac{5}{2}}+\qquad\ldots\qquad(6).$$

The values of p, q, r, s,...are found by substitution in the equation of the curve. The osculating branches will be obtained on approximating to (6). The first approximation gives

$$(y-px)^2=q^2x^3.$$

The next approximation is *not*, $y-px-rx^2=q^2x^3$, but

$$(y-px-rx^2)^2=q^2x^3+{}_2qs\,x^4.$$

Similarly, for further approximations.

Example:

$$ax(y-x)^2=y^4.$$

Putting $y-x=qx^{\frac{3}{2}}+rx^2+sx^{\frac{5}{2}}+\ldots,$

and equating the coefficients of successive powers of x to zero, we get

$$q=a^{-\frac{1}{2}},\; r=2a^{-1},\; s=5a^{-\frac{3}{2}}.$$

Hence $\quad y=x+a^{-\frac{1}{2}}x^{\frac{3}{2}}+2a^{-1}x^2+5a^{-\frac{3}{2}}x^{\frac{5}{2}}+\ldots$

The second approximation to the rational osculating branch is

$$(y-x-2x^2/a)^2=x^3/a+10x^4/a^2.$$

6. The method of division may be employed with equal effect in the case of the osculating branches also. Thus in the example just considered, if we write z for $y-x$,

$$z^2=\frac{(x+z)^4}{ax}=x^3/a+4x^2z/a+6xz^2/a+4z^3/a+z^4/ax.$$

For x^2, in the third term of the right hand expression, put x^3/a, and we get the approximation

$$z^2=x^3/a+4x^2z/a+6x^4/a^2,$$

which is identical with the result already found.

7. If three branches touch one another at the origin, the series for y will be

$$px+qx^{\frac{4}{3}}+rx^{\frac{5}{3}}+sx^2+\ldots\ldots$$

and similar methods apply.

Freedom Equations:

8. Next, let us examine the case of a curve represented by freedom equations.

Suppose $x=f(t)$, $y=g(t)$ are the equations of a curve passing through the origin. To approximate to the form of the curve in the neighbourhood of the origin, we may proceed as follows:

Since the curve passes through the origin, x and y must vanish for the same value of t, say t_1.

Thus $\quad x=(t-t_1)^p\phi(t)\,;\; y=(t-t_1)^q\psi(t).$

Put $t-t_1=z$ and expand by Taylor's Theorem.

$$x=z^p\left\{\phi+z\phi_1+\frac{z^2}{\underline{|2}}\phi_2+\ldots\ldots\right\}$$

$$y=z^q\left\{\psi+z\psi_1+\frac{z^2}{\underline{|2}}\psi_2+\ldots\ldots\right\},$$

suffixes denoting differentiation with respect to t_1. By stopping at different stages and eliminating z, the successive approximations are obtained.

If $p=q$, $y/x=\psi(t_1)/\phi(t_1)$ is the first approximation near the origin.

If $p\neq q$, the first approximation is

$$\left(\frac{x}{\phi}\right)^{q/m}=\left(\frac{y}{\psi}\right)^{p/m}$$

where m is the $g.c.m.$ of p and q. Further approximations may be obtained as usual.

The best method, however, would be to expand z in powers of x first; and substitute the result in

$$\left[\frac{y^{p/m}}{x^{q/m}}\right] = \frac{\{\phi(t_1+z)^{p/m}\}}{\{\psi(t_1+z)^{q/m}\}}$$

by successive approximations.

Examples :

(i) $$x=\frac{at^2}{(1+t)^4};\ y=\frac{at^2}{(1+t)^3}.$$

Here $t_1=0$, and the approximations near the origin are found from $y/x=1+t$, and $x=\dfrac{at^2}{(1+t)^4}$.

Thus $$t=(x/a)^{\frac{1}{2}}(1+t)^2.$$

$\therefore$ $$t=(x/a)^{\frac{1}{2}},\ \text{as the first approximation.}$$

$\therefore$ $$y=x\{1+(x/a)^{\frac{1}{2}}\}\ \text{is the first approximation to the curve.}$$

The second approximation for t is $(x/a)^{\frac{1}{2}}[1+2(x/a)^{\frac{1}{2}}]$, and the second approximation to the osculating branch is therefore

$$y=x\{1+(x/a)^{\frac{1}{2}}+2(x/a)\}$$

The third approximation for t is

$$(x/a)^{\frac{1}{2}}\{1+2(x/a)^{\frac{1}{2}}+5(x/a)\},$$

and the corresponding osculating branch is

$$y=x\{1+(x/a)^{\frac{1}{2}}+2x/a+5(x^3/a)^{\frac{1}{2}}\}.$$

This result agrees with the expansion found in the example of § 5, where the equivalent constraint equation was considered.

(ii) $$x=at^2(1+t);\ y=at^3(1-t).$$

Here the first approximation is evidently

$$ay^2=x^3,$$

or $$y/x=(x/a)^{\frac{1}{2}}.$$

To express y in powers of x, we should consider the relations

$$(y/x)=(x/a)^{\frac{1}{2}}(1-t)(1+t)^{-\frac{3}{2}}$$

obtained by dividing the square of the second equation by the cube of the first. That is

$$y/x=(x/a)^{\frac{1}{2}}\{1-\tfrac{5}{2}t+\tfrac{27}{8}t^2\ldots\ldots\}.$$

Also, from the first equation

$$t=(x/a)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}$$

$$=(x/a)^{\frac{1}{2}}\{1-\tfrac{1}{2}t+\tfrac{3}{8}t^2\ldots\ldots\}$$

Hence, the successive approximations to t are

$$(x/a)^{\frac{1}{2}};\ (x/a)^{\frac{1}{2}}-\tfrac{1}{2}(x/a);\ (x/a)^{\frac{1}{2}}-\tfrac{1}{2}(x/a)+\tfrac{5}{8}(x/a)^{\frac{3}{2}};\ldots\ldots$$

and the corresponding series for (y/x) is

$$(x/a)^{\frac{1}{2}}-\tfrac{5}{2}(x/a)+\tfrac{37}{8}(x/a)^{\frac{3}{2}}-\tfrac{79}{16}(x/a)^2\ldots\ldots$$

The rationalized second approximation is thus

$$(y/x+\tfrac{5}{2}x/a)^2=x/a+\tfrac{37}{4}x^2/a^2.$$

i.e., $$y^2/x^2+5y/a=x/a+3x^2/a^2$$

or $$a^2y^2+5ax^2y=ax^3+3x^4.$$

9. The asymptotes are similarly derived by approximations corresponding to the values of t which make x and y infinite. Suppose t_1 is such a value of t; then we may write

$$x=(t-t_1)^{-p}\phi(t);\ y=(t-t_1)^{-q}\psi(t).$$

Put $t-t_1=z$; and we get

$$x=z^{-p}\,\phi(t_1+z);\ y=z^{-q}\psi(t_1+z).$$

By proceeding as in the previous article the successive asymptotic branches are obtained.

Examples:

(i) $$x=\frac{at^2}{(t-1)^2};\ y=\frac{at^3}{(t-1)^2},$$

Put $t-1=z$; then

$$y/x=1+z.$$

Also, from $x=a(1+z)^2/z^2$, we have

$$z=(a/x)^{\frac{1}{2}}(1+z)$$

By successive approximations

$$z=(a/x)^{\frac{1}{2}}\{1+(a/x)^{\frac{1}{2}}+(a/x)+(a/x)^{\frac{3}{2}}+\ldots\ldots\}$$

$$\therefore\quad (y-x)=\sqrt{ax}+a+a\sqrt{a/x}+a^2/x+\ldots\ldots,$$

which is what we obtained from the constraint equation $x(y-x)^2=ay^2$, in § 1, Ex. (ii).

(ii) $$x=\frac{at^2}{t-1};\ y=\frac{at^2(t+1)}{(t-1)^2}$$

For the asymptote corresponding to $t=1$, put $t-1=z$. Then

$$x=\frac{a(1+z)^2}{z};\ y=\frac{a(1+z)^2(2+z)}{z^2}.$$

$$\therefore \qquad \frac{ay}{x^2}=\frac{2+z}{(1+z)^2}=2(1+\tfrac{1}{2}z)(1-2z+3z^2\ldots\ldots)$$
$$=2\left\{1-\tfrac{3}{2}z+2z^2\ldots\ldots\right\}.$$

Also, from the first equation

$$z=(a/x)(1+z)^2=(a/x)(1+2z+z^2).$$

$$\therefore \qquad z=\frac{a}{x}+2\frac{a^2}{x^2}+5\frac{a^3}{x^3}\ldots,$$ by successive approximations.

Hence

$$\frac{ay}{x^2}=2\left\{1-\frac{3}{2}\left(\frac{a}{x}+\frac{2a^2}{x^2}+\frac{5a^3}{x^3}+\ldots\right)+2\left(\frac{a}{x}+\frac{2a^2}{x^2}+\ldots\right)^2\right\}$$
$$=2-3a/x-2a^2/x^2+\ldots\ldots,$$

i.e.,
$$ay=2x^2-3ax-2a^2+\ldots\ldots$$

This may also be deduced from the constraint equation of the curve $2x^2(y-x)=a(y+x)^2$.

SHORT NOTES.

Notes on Trigonometry.

The method of putting for trigonometrical functions their exponential values can be made use of in solving many examples:

1. Consider the equation :—

$$a\cos2\theta+b\sin2\theta+c\cos\theta+d\sin\theta+e=0. \quad \dots \quad (1)$$

Substituting exponential values and writing x for $\varepsilon^{i\theta}$, we have,

$$x^4(a-ib)+x^3(c-id)+2ex^2+x(c+id)+a+ib=0 \quad \dots \quad (2)$$

Let $\theta_1,\theta_2,\theta_3,\theta_4$ be the roots of equation (1), and let $\theta_1+\theta_2+\theta_3+\theta_4=2s$;

then $$e^{2is}=x_1x_2x_3x_4=\frac{a+ib}{a-ib}, \text{ by (2)}$$

$$\therefore \quad \frac{e^{2is}+1}{e^{2is}-1}=\frac{a}{ib}$$

or $$\frac{a}{\cos s}=\frac{b}{\sin s}=\lambda, \text{ say} \dots \quad \dots \quad \dots \quad (3)$$

$$\frac{\Sigma\{\cos(s-\theta_1)+i\sin(s-\theta_1)\}}{\cos s+i\sin s}=\frac{\Sigma\varepsilon^{i(s-\theta_1)}}{\varepsilon^{is}}=\Sigma\varepsilon^{-i\theta}$$

$$=\Sigma\frac{1}{x_1}=\frac{-(c+id)}{a+ib}=\frac{-(c+id)}{\lambda(\cos s+i\sin s)}$$

$$\therefore \quad \frac{-c}{\Sigma\cos(s-\theta_1)}=\frac{-d}{\Sigma\sin(s-\theta_1)}=\lambda \quad \dots \quad \dots \quad (4)$$

Again

$$\Sigma[\cos\tfrac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)+i.\sin\tfrac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)]$$

$$=\Sigma\varepsilon^{i.\frac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)}=\Sigma\sqrt{\frac{x_1x_2}{x_3x}} \quad \frac{\Sigma(x_1x_2)}{\sqrt{x_1x_2x_3x_4}}$$

$$=\frac{2e}{\sqrt{a^2+b^2}}=\frac{2e}{\lambda}.$$

$$\therefore \quad \Sigma\cos\tfrac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)=\frac{2e}{\lambda}:$$

Hence

$$\frac{a}{\cos s}=\frac{b}{\sin s}=\frac{-c}{\Sigma\cos(s-\theta_1)}=\frac{-d}{\Sigma\sin(s-\theta_1)}=\frac{2e}{\Sigma\cos\frac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)}.$$

[*vide* : Hobson's Trig., 1891, page 84]

A similar method may be used in the consideration of *poristic equations* :

Consider the system of three equations in α,β,γ :

$$a\cos(\beta-\gamma)+b\cos(\beta+\gamma)+c+2f(\sin\beta+\sin\gamma)+$$
$$2g(\cos\beta+\cos\gamma)+2h\sin(\beta+\gamma)=0,$$

and two similar equations.

If the angles α,β,γ be supposed unequal and to lie between 0 and 2π, these three equations are inconsistent unless a certain relation holds connecting the coefficients a,b,c,f,g,h ; and moreover if this relation be satisfied, the three equations are equivalent only to two equations symmetrical in α,β,γ. It is proposed to find these symmetrical equations.

In the above three equations substitute for the trigonometrical functions, their exponential values and for $\varepsilon^{i\alpha}$ write x, for $\varepsilon^{i\beta}$ write y, and for $\varepsilon^{i\gamma}$ write z : we then have

$$a(y^2+z^2)+b(y^2z^2+1)+2cyz-2if[yz(y+z)-y-z]+$$
$$2g[yz(y+z)+y+z]-2ih((y^2z^2-1)=0,$$

i.e. $$a(y^2+z^2)+(b-2ih)y^2z^2+2cyz+2yz(g-if)(y+z)$$
$$+2(g+if)(y+z)+b+2ih=0,$$

and two similar equations ... ... (5)

Subtract the second of these equations from the first and divide by $(y-x)$, since $y \neq x$: we have

$$a(y+x)+(b-2ih)z^2(y+x)+2cz+2(g-if)\{z(y+x+z^2\}$$
$$+2(g+if)=0,$$

and two similar equations. Subtract the second of these from the first and divide by $(-x+z)$, we get

$$-a+(b-2ih)(xy+yz+zx))+2c+2(g-if)(x+y+z)=0, \quad \ldots \quad (6)$$

which is symmetrical in x,y,z, and shows that any one of the equations (5) can be got from the other two. By equating the real and imaginary parts of (6) to zero, we have the two symmetrical independent trigonometrical equations

$$b\Sigma\cos(\beta+\gamma)+2h\Sigma\sin(\beta+\gamma)+2g\Sigma\cos\alpha+2f\Sigma\sin\alpha=a-2c,$$

$$b\Sigma\sin(\beta+\gamma)-2h\Sigma\cos(\beta+\gamma)+2g\Sigma\sin\alpha-2f\Sigma\cos\alpha=0, \quad \ldots \quad (7)$$

equivalent to the original set of three.

To find the relation between the coeffts., denote $\Sigma\cos(\beta-\gamma)$ by $\mathfrak{t}$, and add the three original equations : we get

$$a\mathfrak{t}+b\Sigma\cos(\beta+\gamma)+3c+4f\Sigma\sin\alpha+4g\Sigma\cos\alpha+2h\Sigma\sin(\beta+\gamma)=0.$$

Comparing this with (7) we have,

$$a\mathfrak{t}+a+c+2f\Sigma\sin\alpha+2g\Sigma\cos\alpha=0 \quad \ldots \quad \ldots \quad (8)$$

By squaring and adding equations (7)

$$(b^2+4h^2)[\{\Sigma\cos(\beta+\gamma)\}^2+\{\Sigma\sin(\beta+\gamma)\}^2]=(a-2c)^2-2(a-2c)$$
$$[2g\Sigma\cos\alpha+2f\Sigma\sin\alpha]+4(g^2+f^2)[(\Sigma\cos\alpha)^2+(\Sigma\sin\alpha)^2]$$

i.e, $(b^2+4h^2)[3+2\xi]=(a-2c)^2-2(a-2c)[2g\Sigma\cos\alpha+2f\Sigma\sin\alpha]$
$$+4(g^2+f^2)[3+2\xi],$$

i.e., from (8)

$$(b^2+4h^2)(3+2\xi)=(a-2c)^2+2(a-2c)(a\xi+c+a)$$
$$+4(g^2+f^2)(3+2\xi);$$

or

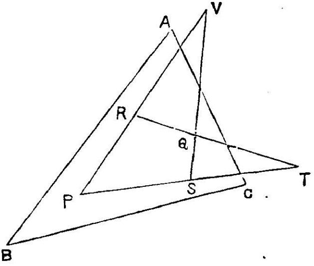
$$2\xi[b^2-a^2+2ac-4(g^2+f^2-h^2)]=-3[b^2-a^2+2ac-4(g^2+f^2-h^2)]$$
$$\therefore \quad b^2-a^2+2ac-4(g^2+f^2-h^2)=0,$$

which is the required relation connecting the coefficients.

E. G. TURNER.

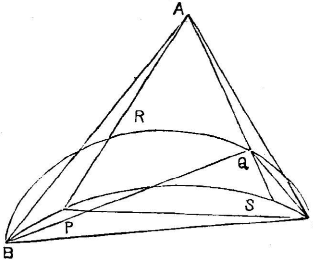
Isogonal Conjugates.

1. Let P, Q and R, S be two pairs of isogonal conjugates of the triangle ABC. Let the opposite sides of the quadrilateral PRQS be produced to meet at V and T. Then V and T will also be isogonal conjugates of the triangle ABC.



For the three pairs of lines from any point to the extremities of the three diagonals of any quadrilateral are in involution. Hence A P, AS; AR, AS and AV, AT are in involution, and AB and AC are conjugate rays of the same pencil. That is, AV and AT make equal angles with AB and AC respectively. Simlarly, with reference to the points B and C; hence V and T are isogonal conjugates.

2. Let P and Q be two isogonal conjugates of the triangle ABC, Join BP, PC and BQ, QC. It is easily seen that $\underline{|BAC} + \underline{|BPC} = \underline{|A} + 180°$



so that, if the angle BPC is constant, BQC is also constant. Therefore if P moves on the circle APC, Q will move on the circle BQC.

Now, suppose AP and AQ meet the circles BQC and BPC at R and S. Then R and S are isogonal conjugates. Therefore RQ and PS will meet at a point T which will be the isogonal conjugate of the point A by 1. Hence the point T is on the base BC, which is the radical axis of the two circles. Hence the four points, P,Q,R, S are concyclic:

3. In trilinear co-ordinates with ABC as the triangle of reference, the equation of the circle BPC will be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + \lambda\alpha(a\alpha + b\beta + c\gamma) = 0. \quad \ldots \quad \ldots \quad (1)$$

Then the circle BQC will be

$$\frac{a}{\beta\gamma} + \frac{b}{\gamma\alpha} + \frac{c}{\alpha\beta} + \frac{\lambda}{\alpha}\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right) = 0.$$

i.e.,

$$a\alpha + b\beta + c\gamma + \frac{\lambda}{\alpha}(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0$$

or

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + \frac{\alpha}{\lambda}(a\alpha + b\beta + c\gamma) = 0 \quad \ldots \quad (2)$$

If P is $(\alpha_1, \beta_1, \gamma_1)$; and Q is $(\alpha_2 \beta_2 \gamma_2)$ and $\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2 = k$,

then

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 2\triangle = \left(\frac{a}{\alpha_2} + \frac{b}{\beta_2} + \frac{c}{\gamma_2}\right)k.$$

Hence

$$a\beta_2\gamma_2 + b\gamma_2\alpha_2 + c\alpha_2\beta_2 = 2\triangle\alpha_2\beta_2\gamma_2/k.$$

Similarly

$$a\beta_1\gamma_1 + b\gamma_1\alpha_1 + c\alpha_1\beta_1 = 2\triangle\alpha_1\beta_1\gamma_1/k.$$

Applying these results to (1) & (2), we get

$$2\Delta\alpha_1\beta_1\gamma_1/k = -2\Delta\lambda\cdot\alpha_1.$$
$$2\Delta\alpha_2\beta_2\gamma_2/k = -2\Delta\alpha_2/\lambda.$$

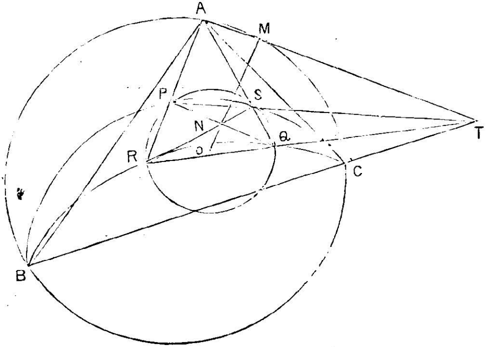
$\therefore$ $$\beta_1\gamma_1 : \beta_2\gamma_2 = \lambda^2.$$

But $\beta_1\beta_2 = \gamma_1\gamma_2$; hence $\beta_1 : \gamma_2 = \gamma_1 : \beta_2 = \lambda$.

Therefore AP : AQ is constant so long as P is on the circle BPC; and AP.AR = AQ AS. Hence RQ, PS meet on BC.

Also, if ABC is any triangle on a given base BC having a given vertical angle, the locus of the isogonal conjugates of the triangle ABC whose distances from A are in a given ratio are two circles. The two become one when the ratio is unity.

4. Let TA meet the circle ABC at M. Then M is the double point of directly similar figures on (PS, RQ) or (PR, SQ).



For, the double point in question is the inverse of N (the intersection of PQ, RS) with respect to the circle PSQR and therefore lies on AT, which is the polar of N with respect to the circle PSQR The double point also lies on the circumcircle ABC. Hence, the result.

10*th August*, 1912. T. Rajarama Rao.

The Face of the Sky for November and December 1912.

(The time referred to is Standard time).

Sidereal time at 8 p.m.

	November.				*December.*		
	H.	M.	S.		H.	M.	S.
1	22	42	2	...	0	40	19
8	23	9	38	...	1	7	55
15	23	37	14	...	1	35	31
22	0	4	49	...	2	3	7
29	0	32	25	...	2	30	43

From this table the constellations visible during the evenings can be ascertained by a reference to their position as given in the star-charts.

Phases of the Moon.

	D.	H.	M.			D.	H.	M.	
Last Quarter ...	2	9	8	A.M.	...	1	4	55	P.M.
New Moon ...	9	7	35	,,	...	8	11	7	,,
First Quarter ...	17	4	13	,,	...	17	2	7	A.M.
Full Moon ...	24	9	42	P.M.	...	24	10	0	,,
Last Quarter ...	...	...	...		...	31	1	42	,,

The Sun.

The Sun is in Winter Solstice on December 22nd at 10-15 A.M.

The Planets.

Mercury begins as an evening star, attains its greatest elongation, 22°14'E. on November 19 at 6-30 p.m., becomes stationary on November 29 and is in inferior conjunction with the Sun on December 9, when it becomes a morning star. It is again stationary on December 19 and attains its greatest elongation, 22°23' W on December 28 It is in conjunction with the Moon on November 11 and December 9, and with Jupiter on November 21 and December 3.

Venus continues an evening star. It is in conjunction with the Moon on November 12 and December 12, and with Jupiter on November 8, and with Uranus on December 13.

Mars is in conjunction with the Sun on November 5, and with the Moon on November 9 and December 8.

Pallas is in conjunction with the Moon on November 6 at 2-30 a.m.

Jupiter is in conjunction with the Sun on December 19 and with the Moon on November 11 at 6-51 p.m. and December 9.

Saturn is in opposition to the Sun on November 23. It is in conjunction with the Moon on November 24 at 9-17 p.m. and on Dec. 22.

Uranus is in conjunction with the Moon on Nov. 15 at 1-29 a.m. and on December 12. The position of the planet at 8 p.m. on November 15 is

R. A.	Decl.
20^h 9^m 41^s.	20° 42′ 9″ S.

Neptune is in conjunction with the Moon on Nov. 28 and Dec. 25 at 9-54 p.m. The position of the planet at 8 p.m. on December 15 is

R. A.	Decl.
7^h 49^m 11^s.	20° 33′ 31″ N.

V. RAMESAM.

REVIEWS.

L'Equation de Fredholm et ses applications a' la physique mathematique : par M. M. H. B. Heywood et M. Fre'chet.

(A. Hermann & Fils, Paris, 1912).

The theory of integral equations has been, in recent years, one of the favourite subjects of mathematical investigation. The first integral equation was met with in a physical problem as far back as 1823. The systematic study, however, of this kind of equations is of a quite recent date. Many of the brilliant results arrived at by mathematicians are of great importance to the physicist as was to be expected from a branch of analysis which made its first appearance in physics.

The book under review gives a clear and good summary of such portions of the theory as may be regarded to have taken a definite shape and have an important bearing on mathematical physics. It is not the aim of the authors to write a complete treatise. Of the three kinds of integral equations only the second, is dealt with. This equation has the form :—

$$\phi(s) - \lambda \int_a^b K(s,t)\,\phi(t)\,dt = f(s),$$

where $\phi(s)$ is the unknown function, λ a parameter, and $K(s, t)$, called the kernel, and $f(s)$ are two known functions.

In the short introduction a number of important definitions and propositions are given. In the first Chapter the physical problems to be dealt with are explained and the equations transformed into integral equations. These problems fall under the two headings :—

1. Problems connected with the equation $\Delta V = 0$,

2. Those connected with the equation $\Delta V = R(x,y,z)\, V + R_1(x,y,z)$.

The second Chapter is of a general character and gives the different methods of solving Fredholm's equation. In the concluding third Chapter these methods are applied to the problems introduced in Chapter I. For further details we must refer our readers to the excellent work itself.

In the bibliography at the end, the pamphlet of R. d' Adhe'mar : *L' Equation de Fredholm et les problemes de Dirichlet et de Neumann*, might have been mentioned.

The book is well written, clear, and elementary. The reader is supposed to be acquainted with the partial differential equations of physics. We strongly recommend the study of the book to all who take an interest in mathematical physics.

6th June, 1912. A. STEICHEN.

Engineering Applications of Higher Mathematics, Part I (*Problems on Machine Design*): by V. Karapetoff, small 8vo., xiv + 69, John Wiley & Sons, New York, 1912; 75 cents.

This is a unique publication. There are several advanced treatises on the different branches of engineering where a liberal use is made of higher mathematics. But there are only very few works where the application of analytical geometry or calculus to engineering problems is considered. The book under review is, intended to enable an engineer to make a better and more extended use of higher mathematics in his work. With this object in view, the author selects some machine element as the subject of each chapter, takes up for investigation such aspects of the element as require for their determination the aid of the calculus and the analytical conics, and by close mathematical reasoning arrives at formulæ for the determination of these aspects. The remainder of each chapter consists of a number of valuable examples where the assumed data of the machine element are varied on several iractical considerations.

In the first chapter, the author devotes himself to the determination of the angle of inclination of an inclined plane (used as a mechanical device) that will give the maximum efficiency for a given angle of friction. A square-threaded screw is dealt with in this chapter as a special case of an inclined plane over a cylindrical surface. The second chapter is devoted to the finding of the general expression by integration, of the total amount of friction in a circular bearing, and various examples assuming different methods of distribution of pressure in the different parts of the bearing complete this chapter. Chapter III is similarly devoted to the determination of friction in step bearings of various designs. In Chapter IV is found, by integration, the relation between the tensions on the driving and slack sides of a belt transmitting power, and various ingenious problems follow which throw a flood of light on several points affecting the design of belts and pulleys. Chapter V deals with the strength of shafts subject to torsion. The problems propounded here are based on various practical considerations to find the relative advantages of hollow and solid shafts. In Chap-

ter VI, the author proceeds to calculate the energy in a flywheel and finds that it depends on its moment of inertia. There is a number of interesting problems in this chapter bearing on flywheels of various rims and spoke sections.

The problems in all the chapters are ingeniously devised, and being based on practical considerations affecting the design of the several machine elements are well calculated to develop the reader's power of application of higher mathematics to engineering problems. The author has been successful in securing the object of his publication.

The book is prefaced with a parable describing humourously the manner in which engineering and mathematics are taught in engineering colleges completely divorcing the one from the other. The book is intended to remedy this defect and should, in the hands of students, make a useful text-book to link together his mathematical and engineering studies.

There is yet another feature of the work which deserves notice. The book serves as an eye-opener to the many engineers in India as to how much they could improve the quality of their work by an application of the principles of mathematics in their every-day problems and as to how little they generally think of them. Unfortunately, the average engineer in India forgets most of his mathematics before he rises high in his profession, and when his professional duties demand original investigation in which a knowledge of mathematics is required he has to repent his past neglect. The truth is that the arduous duties of an engineer give him very little leisure to keep up his mathematical studies. But it behoves him to find the leisure for the purpose ; for, apart from the intellectual pleasure derived, the engineer's mind cannot be at ease unless he is satisfied that the designs he makes in the discharge of his professional duties are on a strictly scientific basis.

The book is priced at only 75 cents and should find a place in every engineer's library. It can be had of the illustrious firm of publishers : Messrs. John, Wiley and Sons of New York, or Messrs. Chapman and Hall, London.

The author proposes to publish similar volumes of problems taken from Hydraulics and Strength of Materials, which should prove even more interesting and useful to the civil engineer.

A. V. K. Aiyar.

Calculus for Beginners, by **W. M. Baker, M.A.** (G. Bell and Sons : Ld.) Price 3/-

This little volume of 166 pages contains seventeen small chapters. The first six chapters deal with differentiation and the remaining eleven with integration. In differentiation only $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are considered, the former for the determination of the maxima and the minima values of y and the latter for the determination of the points of inflexion on a curve. The idea of a point of inflexion on a curve is well impressed on the mind of the beginner by defining the point as one at which $\frac{dy}{dx}$ has a *turning value*. In one of the chapters the differential coefficient as a rate measurer is clearly explained and the expressions $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ for velocity and acceleration are obtained and illustrated. The author might have done a greater service to beginners by introducing two more small chapters in this part, one on asymptotes and the other on curve-tracing. The second part of the book is fully treated as far as beginners are concerned. The relation of $\int f'(x)\,dx$ to $f(x)$ is clearly explained. There are small chapters on integration by parts, the determination of areas, volumes, lengths of arcs, centre of gravity, work, moment of inertia and centre of pressure. There is also a small chapter on differential equations, with a short account of the simple harmonic motion.

The book should be in the hands of every student preparing for the B.A. (Pass Course) in Groups I and II-A. of the University of Madras.

S. P. S.

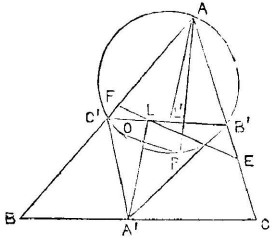
SOLUTIONS.

Question 275.

(M. Bhimasena Rao) :—Shew that the foci of the three parabolas each touching two of the three sides of a triangle and also the join of their middle points at the point where this join is intersected by the corresponding side of the orthocentric triangle, are collinear.

Remarks by the Proposer.

Let O be the circumcentre of the triangle ABC ; A′B′C′ the medial triangle ; E, F, the feet of the perpendiculars from B, C ; L, the point intersection of B′C′ and EF ; L′, the isotomic conjugate of L with respect to B′C′ ; and P, the focus of the parabolad escribed to AB′C′ touching B′C′ at L.



Then A′L passes through the Steiner point of A′B′C′. But L and L′ are corresponding points of the homothetic triangles A′B′C′ and AC′B′. Therefore AL′ passes through the Steiner point of AB′C′.

Also AL′ is a diameter of the parabola ; and AP is the isogonal conjugate of AL′ with respect to the angle B′AC′. Therefore the pedal line of the Steiner point of AB′C′ is perpendicular to AP and therefore parallel to OP, since OPA is a right angle. We know that the pedal line of the Steiner point is parallel to the Brocard line. Therefore OP and the Brocard line of AB′C′ are parallel, and hence are corresponding lines of ABC and AB′C′. Therefore OP is the Brocard line of ABC.

Hence, the focus P lies on the Brocard line of ABC, and is the projection of A on it.

Similarly, the foci of the remaining parabolas are the projections of B and C on the Brocard line of ABC.

Question 284.

(S. RAMANUJAN) :—Solve

$$\frac{x^5-6}{x^2-y}=\frac{y^5-9}{y^2-x}=5(xy-1).$$

Solution by the Proposer.

Let $x=\alpha+\beta+\gamma$, and $y=\alpha\beta+\beta\gamma+\gamma\alpha$, where α,β,γ may have any values we choose. By substituting the supposed values of x and y in the given equations, we find that $\alpha\beta\gamma$ should have the value 1.

Now $(\alpha+\beta+\gamma)^5-6=5\ \{(\alpha+\beta+\gamma)^2-(\alpha\beta+\beta\gamma+\gamma\alpha)\}$
$$\times\ \{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma\}.$$

and $(\alpha\beta+\beta\gamma+\gamma\alpha)^5-9=5\{(\alpha\beta+\beta\gamma+\gamma\alpha)^2-\alpha\beta\gamma(\alpha+\beta+\gamma)\}$
$$\times\{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)\alpha\beta\gamma-\alpha^2\beta^2\gamma^2\}.$$

Simplifying
$$\alpha^5+\beta^5+\gamma^5=6,$$
$$(\alpha\beta)^5+(\beta\gamma)^5+(\gamma\alpha)^5=9.$$

Also
$$(\alpha\beta\gamma)^5=1.$$

Thus we see that α^5, β^5 and γ^5 are the roots of the equation
$$t^3-6t^2+9t-1=0.$$

But it is easily seen that the roots of this equation are
$$4\cos^2 20°,\ 4\cos^2 40°\ \text{and}\ 4\cos^2 80°.$$

Hence
$$\alpha=(2\cos 20°)^{\frac{2}{5}},\beta=(2\cos 40°)^{\frac{2}{5}},\gamma=(2\cos 80°)^{\frac{2}{5}},$$
$$x=(2\cos 20°)^{\frac{2}{5}}+(2\cos 40)^{\frac{2}{5}}+(2\cos 80)^{\frac{2}{5}},$$

and
$$y=(\tfrac{1}{2}\sec 20)^{\frac{2}{5}}+(\tfrac{1}{2}\sec 40)^{\frac{2}{5}}+(\tfrac{1}{2}\sec 80)^{\frac{2}{5}}.$$

All the values of x are
$$\alpha+\beta+\gamma,\ \alpha+\beta\rho+\gamma\rho^4,\ \alpha+\beta\rho^2+\gamma\rho^3,\ \alpha\rho+\beta\rho+\gamma\rho^3,$$
$$\alpha\rho+\beta\rho^2+\gamma\rho^2,\ \alpha\rho^2+\beta\rho^4+\gamma\rho^4,\ \alpha\rho^3+\beta\rho^3+\gamma\rho^4,$$

where ρ is an imaginary fifth root of unity.

Similarly for y.

Question 298.

(M. T. NARANIENGAR, M.A.) :—Find the parabolic asymptote of the curve $(y-x)^2(y+x)=2ax^2$, and discuss the correctness of the solution given on p. 201, Edward's *Diffl. Calc.*, 1896.

[For solution see p. 163 of this number.]

Question 312.

(Jamshedji Edalji) :—A Japanese chessboard has 81 cells: they are supposed to be black and white alternately. Determine a route by which a knight starting from a given cell occupies successively *all* the cells, *once* and *only once* and terminates its path in another given cell—both the given cells having the same colour as the four corner cells of the chessboard.

Solution by the Proposer.

The path excluding the innermost cell may be divided into four equal parts, the cells of each part being arranged in two circuits. By turning the chess-board round we can make any one of the four parts coincide with any other part.

B_3	H_8	B_9	H_2	B_5	C_2	C_7	E_8	E_3
B_8	H_3	B_4	H_7	B_{10}	E_7	E_2	C_3	C_8
H_9	B_2	H_5	B_6	H_1	C_6	C_1	E_4	E_9
H_4	B_7	H_{10}	B_1	H_6	E_1	E_6	C_9	C_4
A_5	A_{10}	G_5	G_{10}	I	C_{10}	C_5	E_{10}	E_5
G_4	G_9	A_6	A_1	D_{10}	F_1	D_6	F_7	D_2
A_9	A_4	G_1	G_6	D_5	F_6	D_1	F_2	D_7
G_8	G_3	A_2	A_7	F_{10}	D_9	F_4	D_3	F_8
A_3	A_8	G_7	G_2	F_5	D_4	F_9	D_8	F_3

The problem can therefore be completely solved by commencing on the cells A_1, A_3, A_5, A_7, A_9, G_1, G_3, G_5, G_7, G_9, or by commencing on the cells A_1, A_3, A_5, A_7, A_9, H_1, H_3, H_5, H_7, H_9. The cells of the A group form a path only if we commence or end on A_3, A_5, A_8, or A_{10}, or

if we commence and end on A_1 and A_2, or on A_6 and A_7. The following are the eight paths not shown in the figure :—

(1). $A_3A_2......A_7A_4A_5A_6$.
(2). $A_3A_2A_9A_{10}A_1A_4......A_8$.
(3). $A_3A_2A_1A_4......A_{10}$.
(4). $A_5A_6......A_1A_4A_3A_2$.
(5). $A_5A_6A_7A_4A_3......A_8$.
(6). $A_5......A_9A_2A_3A_4A_1A_{10}$.
(7). $A_1A_{10}A_9A_2......A_8$.
(8). $A_7A_8A_9A_6A_5......A_1A_{10}$.

The cells of the H group form a path only if we commence or end on H_1, H_9, H_4, or H_6, or if we commence and end on H_2 and H_3, or on H_7 and H_8. The following are the eight paths of the H group not shown in the figure :—

(1). $H_1H_{10}......H_5H_2H_3H_4$.
(2). $H_1......H_5H_8H_9H_{10}H_7H_6$.
(3). $H_1......H_7H_{10}H_9H_8$.
(4). $H_9H_8......H_3H_{10}H_1H_2$.
(5). $H_9......H_5H_2H_1H_{10}H_3H_4$.
(6). $H_9H_8H_7H_{10}......H_6$.
(7). $H_7H_6H_5H_8H_9......H_4$.
(8). $H_3H_4H_5H_2......H_6$.

If we begin and end on any two cells of the A group, we may split it up into two parts as follows: the first part ending on A_6 or A_{10} and the second part beginning on A_{10} or A_8 :—

(1). $A_3A_2A_9A_8A_7A_4A_5A_6$ and $A_{10}A_1$.
(2). A_5A_6 and $A_{10}......A_7A_4......A_1$.
(3). $A_1A_2......A_6$ and $A_{10}......A_7$.
(4). A_9A_{10} and $A_8......A_1$.
(5). A_5A_6 and $A_{10}......A_7A_4A_1A_2A_8$.
(6). $A_3A_2A_1A_4A_5A_6$ and $A_{10}......A_7$.
(7). $A_9A_8A_7A_4A_5A_6$ and $A_{10}......A_3$.
(8). $A_7A_8A_9A_6$ and $A_{10}......A_5$.
(9). $A_9......A_6$ and $A_{10}......A_5$.
(10). $A_7A_6......A_{10}$ and A_8A_9.

The above arrangements can be easily remembered if it be borne in mind that A_4 may be followed or preceded by A_1 or A_7 and A_9 may be followed or preceded by A_8 or A_2.

If we begin and end on any two cells of the H group, we may break it up into two parts as follows: the first part ending on H_6 and the second part beginning on H_4 or H_{10} :—

(1). $H_1H_{10}......H_6$ and $H_4H_5H_2H_3$.
(2). $H_1H_{10}......H_6$ and $H_4H_8H_2H_5$.
(3). $H_1......H_6$ and $H_{10}......H_7$.
(4). $H_1......H_6$ and $H_{10}H_7H_8H_9$.
(5). $H_3H_2......H_6$ and H_4H_5.
(6). H_7H_6 and $H_4H_5H_8......H_3$.
(7). $H_9......H_6$ and $H_{10}H_1H_2H_5H_4H_3$.
(8). H_5H_6 and $H_4H_3......H_7$.
(9). $H_9......H_6$ and $H_{10}H_1......H_5$.
(10). H_7H_6 and $H_{10}H_1......H_5H_8H_9$.

The above arrangements can be easily remembered by noting that H_5 may be followed or preceded by H_2 or H_8, and H_{10} may be followed or preceded by H_7 or H_3.

Denoting the parts into which the cells of the A and H groups are divided by a_1, a_2, and h_1, h_2, we have the following five arrangement. which give a complete solution of the problem :—

(1) a_1 (or $h_1A_1A_{10}$*) $B_1B_{10}C_1C_{10}D_5D_6$ I $E_6E_5F_2F_1H_6H_5G_{10}G_1a_2$ (or $F_2F_1G_{10}G_9h_2$).

(2) AA_{10}† H_9H_{10} (or $HH_6A_1A_2$) $G_9G_{10}F_1F_2E_5E_6$ I $D_6D_5C_{10}C_1B_{10}B$ (or $B_6B_5C_6C_5D_{10}D$).

(3) $AA_{10}H_9H_{10}$ (or $HH_6A_1A_2$) $G_9G_{10}B_7B_6$ I $F_6F_7E_6E_5D_6D_5C_{10}C$ (or $C_4C_5D_{10}D_1E_{10}E$),

(4) $AA_{10}H_9H_{10}$ (or $HH_6A_1A_2$) $G_9G_{10}B_7B_6E_7E_6$ I $C_6C_5D_{10}D_1F_{10}F$ (or $F_1F_{10}D_1D_{10}C_5C_6I$).

(5) AA_{10} (or $HH_6A_1A_{10}$)$B_1B_{10}C_1C_{10}D_5D_6$ I $E_6E_5F_2F_1G_6G_5H_6H$ (or $H_6H_5G_{10}G$ or $G_{10}G$).

Other Solutions by N. B. Pendse, N. Sankara Aiyar, B.A. and M. Satyanarayana, M.A.

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* A_1A_{10} denotes the circuit from A_1 to A_{10} : D_5D_6 denotes the circuit $D_5D_4......D_6$.

† AA_{10} denotes a path of the cells of the A group ending in A_{10}. $A_{10}A$ denotes a path of the cells beginning with A_{10}. BB_{10}, $B_{10}B$, &c., have similar meanings.

Question 325.

(R. SRINIVASAN, M. A.) :—

If $a_n = \int_0^1 t\,(t-1)\,(t-2)\ldots\ldots(t-n+1)\,dt$, shew that

$$\sum_0^\infty (-1)^m\, a_m \frac{\lfloor x-1}{\lfloor x+m} = \log\left(1+\frac{1}{x}\right).$$

Solution by S. Ramanujan.

$$\sum_0^\infty (-1)^m\, a_m \frac{\lfloor x-1}{\lfloor x+m}$$

$$= \sum_0^\infty \left[\int_0^1 (-1)^m.\ \frac{t\,(t-1)\,(t-2)\ldots\ldots(t-m+1)}{x\,(x+1)\,(x+2)\ldots\ldots(x+m)}\,dt\right]$$

$$= \int_0^1 \left\{\frac{1}{x} - \frac{t}{x\,(x+1)} + \frac{t\,(t-1)}{x(x+1)\,(x+2)} - \&c.\right\} dt$$

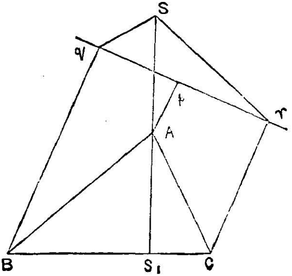
$$= \int_0^1 \frac{dt}{t+x} = \log\left(1+\frac{1}{x}\right).$$

Question 328.

(M. BHIMASENA RAO) :—Show that the normal coordinates of the orthopole of $p\alpha+q\beta+r\gamma=0$, are
$(-p+q\cos C+r\cos B)\ (-p\cos B\cos C+q\cos B+r\cos C)$. etc.

Solution by W. Gallatly.

In this solution p, q, r are the $\perp$ *rs* on the line from A,B,C ; the *acute* angles made by the line with BC,CA,AB are $\theta_1\theta_2\theta_3$; the orthopole is S.



It is known that

$$2\,\Delta\cos\theta_1 = ap - bq\cos C - cr\cos B.$$

The angles Spq, qSp are $90^\circ-\theta_1$ and C ;

also $$pq = c\cos\theta_3:$$

$$\therefore\quad Sq = 2R\cos\theta_3\cos\theta_1$$

$$\therefore\quad \alpha = SS_1 = \text{projection of } B\,q\,S \text{ on } SS_1$$
$$= q\cos\theta_1 + 2R\cos\theta_3\cos\theta_1\cos C.$$

$$\therefore\quad \frac{\alpha}{\cos\theta_1} = q + 2R\cos C\,(cr - ap\cos B - bq\cos A)\,/2\Delta.$$

$$\therefore\quad \frac{2\,\Delta\,\alpha}{4R^2\cos\theta_1} = \sin A\sin C\; q\sin B + \cos C\; r\sin C.$$
$$-\cos B\cos C\cdot p\sin A - \cos A\cos C\cdot q\sin B.$$
$$= \cos B\cdot q\sin B + \cos C\cdot r\sin C - \cos B\cos C\cdot p\sin A$$

$$\therefore\quad 2\Delta^2\,\alpha = R\,(pa - q\,b\cos C - rc\cos B\cdot)\times$$
$$(\cos B.\ qb + \cos C.\ rc - \cos B\cos C\cdot pa),$$

giving the *absolute* normal co-ordinates of the orthopole.

Multiplying the factors, and using the relation

$$\Sigma\, a^2p^2 - 2.\Sigma bq\cdot cr.\ \cos A = 4\,\Delta^2,$$

we find $2\,\Delta\cdot\alpha = q(r-p)\,c\cos B - r(p-q)\,b\cos C + abc\cos B\cos C,$

(see Gallatly's Geometry, section 79.)

When the line cuts the circumcircle ABC, say in T,T′, the simson lines of these points intersect at the orthopole: and if their direction angles are $\lambda\mu\nu$, $\lambda'\mu'\nu'$, then $\alpha = 2R\cos\theta_1\sin\lambda\sin\lambda'$: (Geometry, section 72.)

Hence $2\,\Delta\,\sin\lambda\sin\lambda' = qb\cos B\cdot + rc\cos C\cdot - pa\cos B\cos C.$

Question 361.

(J. C. Swaminarayan, M.A.) :—Prove that

$$a\,(b-c)\,(b+c-a)\,(b^2+c^2-a^2) + \text{two } \textit{similar} \text{ expressions} = 0.$$

Solution by the Proposer.

If a,b,c are the sides of a triangle ABC,

$$4\sin^2\frac{B}{2} - 4\sin^2\frac{C}{2}$$
$$= \frac{(a+b-c)\,(b+c-a)}{ca} - \frac{(b+c-a)\,(c+a-b)}{ab}$$
$$= \frac{b+c-a}{a}\left\{\frac{a+b-c}{c} - \frac{c+a-b}{b}\right\}$$
$$= \frac{b+c-a}{a}\left\{\frac{a\,(b-c)+(b^2-c^2)}{bc}\right\}$$
$$= \frac{(b+c-a)(b-c)\,(a+b+c)}{abc}$$

Now $\Sigma\, 4 \cos A \left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)$

$$= \Sigma\, 4\left(1-2\sin^2\frac{A}{2}\right)\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)=0$$

$$\therefore \quad \Sigma\frac{(b^2+c^2-a^2)(b+c-a)(b-c)(a+b+c)}{2\,bc\times abc}=0.$$

$$\therefore \quad \Sigma\frac{(b-c)(b+c-a)(b^2+c^2-a^2)}{bc}=0$$

$$\therefore \quad \Sigma\, a\,(b-c)\,(b+c-a)\,(b^2+c^2-a^2)=0.$$

[The question was wrongly printed with $(b^3+c^2-a^3)$ as the last factor. Mr. T. P. Trivedi, M.A., LL.B. reads $(b^3+c^3-a^3)$, and proves that the sum is $2\,abc\,(a-b)\,(b-c)\,(c-a)$. Ed.]

Question 362.

(S. Narayan, B.A., L.T.) :—ABC is a triangle whose symmedian point is K, PQR is the reciprocal polar of ABC with respect to K, the modulus being the power K with respect to the circumcircle. Shew that the polar conic of K with respect to PQR is the locus of points whose pedal triangles with respect to ABC, have their centroids on the circumcircle of ABC.

Solution by M. Bhimasena Rao, and J. C. Swaminarayan, M.A.

Taking ABC as the triangle of reference, if the normal co-ordinates of any point O be l, m, n, the mean centre of the vertices of the pedal triangle of O for the multiples, p, q, r is $l(q+r)+qm\cos C+rn\cos B$, &c.

Let PQR be the triangle formed by the lines

$$\alpha(q+r)+q\,\beta\cos C+r\,\gamma\cos B=0 \quad \ldots \quad \ldots (1)$$

$$\beta(r+p)+r\,\gamma\cos A+p\,\alpha\cos C=0 \quad \ldots \quad \ldots (2)$$

$$\gamma(q+p)+p\,\alpha\cos B+q\,\beta\cos A=0 \quad \ldots \quad \ldots (3)$$

The areal co-ordinates of l, m, n with respect to PQR are

$$\{\,l(q+r)+q\,m\cos C+r\,n\cos B\,\}\times\begin{vmatrix} a & b & c \\ p\cos C, & (r+p), & r\cos A \\ p\cos B, & q\cos A, & p+q \end{vmatrix}, \text{ etc.,}$$

i.e., $a\,\{\,l(q+r)+qm\cos C+rn\cos B\,\}$, etc.,

on simplifying the determinants and removing the common factor $pq\sin^2 C+qr\sin^2 A+rp\sin^2 B$.

The mean centre and the point O have the same co-ordinates, in areals, with reference to the triangles ABC and PQR respectively. Therefore, when the mean centre moves on any curve, O moves on the corresponding curve of PQR.

From (1), (2), (3) we see that ABC and PQR are conjugate with respect to the curve

$$qr(\beta^2+\gamma^2+2\,\beta\gamma\cos A)+\&c.,=0.$$

This is a circle cutting orthogonally the circumcircle of ABC and having its centre at a/p, b/q, c/r. (see page 55. Vol. II, J. I. M. S.)

The double point of ABC and PQR is a/p, b/q, c/r the centre of the circle, by Question 367. The polar conics of the double point with respect to ABC and PQR are corresponding curves. Therefore when the mean centre moves on the polar conic of the double point with respect to ABC, the point O moves on the polar conic of the double point with respect to PQR.

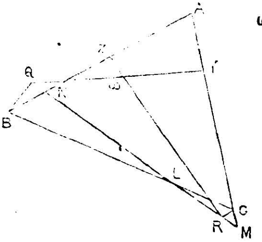
Putting $p=q=r$, we obtain the required result.

Question 366.

(M. KANNAN, B.A., L.T.) :—In a triangle ABC, PP′ is a diameter of the circumcircle, and the Simson lines of P, P′ intersect at ω. Show that the length of the perpendicular from ω on PP′ is $2\,R\cos\alpha\cos\beta\cos\gamma$, where α, β, γ are the inclinations of PP′ to the sides of the triangle.

Solution (1) by M. Bhimasena Rao, (2) by G. Ramachandran.

(1) Let ABC, be the triangle. From B and C draw BQ and CR perpendicular to any line LMN from Q and R draw QY and RZ perpendicular to CA and AB respectively meeting in ω. Then ω is the point of intersection of the pedal lines of the points in which LMN meets the circumcircle of ABC.



In the triangle ωQR, the perp. from ω on QR

$$=\frac{QR}{\sin Q\omega R}.\ \sin\omega QR.\ \sin\omega RQ.$$

$$=\frac{a}{\sin A}.\ \cos\alpha\ \cos\beta\ \cos\gamma,$$

$$=2\,R\cos\alpha\cos\beta\cos\gamma.$$

(2) The A′B′C′ Simson line of ω makes angles α, β, γ with the sides, and is therefore parallel to POP′. This Simson line bisects ωO′ since O is the ortho centre of A′B′C′. [See: Article on Simson line by W. Gallatly in August 1910, J. I. M. S., § 16.]

Therefore the simson line of ω bisects the perpendicular from ω on POP′.

The perpendicular from ω on the A′B′C′ simson line is equal to R $\cos\alpha \cos\beta \cos\gamma$, since the diameter of A′B′C′ is the radius of ABC. [*Ibid*, § 6.]

Hence, $\perp^r$ from ω on POP′ $= 2$ R $\cos \alpha \cos \beta \cos \gamma$.

Question 370.

(K. J. Sanjana, M. A.) :—Solve graphically the equation

$$a \tan\theta + b \sin \theta = c,$$

where a, b, c are numerical constants.

Solution by J C. Swaminarayan, M.A.

Draw the hyperbola $x^2/b^2 - y^2/a^2 = 1$.

The normal at the point (b cosec θ, $a \cot \theta$) is

$$\frac{(x - b \operatorname{cosec}\theta)b}{\operatorname{cosec}\theta} + \frac{(y - a \cot\theta)a}{\cot\theta} = 0,$$

i.e. $$bx \sin\theta + ay \tan\theta = a^2 + b^2.$$

This normal will pass through the point $\left(\frac{a^2+b^2}{c}, \frac{a^2+b^2}{c}\right)$, if

$$a \tan\theta + b \sin\theta = c,$$

which is the equation proposed.

Hence, we should draw normals from the point $(a^2+b^2)/c$, $(a^2+b^2)/c$ to the hyperbola $x^2/b^2 - y^2/a^2 = 1$; and the solutions of the given equation correspond to the four feet P_1, P_2, P_3, P_4.

Thus, if P_1N_1 be perpendicular to the transverse axis and from N_1 a tangent be drawn to the auxiliary circle, the angle θ_1 will be equal o the angle between the tangent and the transverse axis. Similarly tthe angles θ_2, θ_3, θ_4 are obtained.

Question 372.

(D. D. Kapadia, M.A., B. Sc.) :—A rectangular hyperbola has contact of the highest order possible with a given curve. Prove that the semi-transverse axis of the hyperbola is

$$\rho\left\{1 + \frac{1}{9}\left(\frac{d\rho}{ds}\right)^2\right\}^{-\frac{3}{4}}$$

Solution (1) by G. Ramachandran and V. K. Aravamudan, and (2) by T. P. Trivedi, M.A., LL.B.

(1) The equation of a rectangular hyperbola is written $r^2\cos 2\theta = a^2$, where a is the semi-transverse axis. With the usual notation

$$\phi = \pi/2 - 2\theta,\ \psi = \phi + \theta = \pi/2 - \theta.$$

$$\therefore \quad \rho = ds/d\psi = -ds/d\theta = -r\ \mathrm{cosec}\phi = -a\sec 2\theta^{\frac{3}{2}}.$$

$$\therefore \quad 1/\rho .\ d\rho/d\theta = 3\tan 2\theta$$

i.e. $\quad d\rho/ds = -3\tan 2\theta$, since $\rho d\theta = -ds$.

Hence $\quad a = -\rho\cos^{\frac{3}{2}} 2\theta = -\rho\left\{1 + \frac{1}{9}\left(\frac{d\rho}{ds}\right)^2\right\}^{-\frac{3}{4}}$

(2) Referring the hyperbola to the common tangent and normal as axes, its equation may be taken

$$2y = a(x^2 - y^2) + 2hxy.$$

Expanding y by Maclaurin's theorem,

$$y = px + \frac{qx^2}{\underline{|2}} + \frac{rx^3}{\underline{|3}} + \ldots\ldots$$

where p, q, r are the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, at the origin. These are known, as the contact is of the third order. Then, as in Edward's Diff. Calculus, pages 281-82, we get

$$p = 0,\ a = q,\ qh = \frac{r}{3};\ \rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{1}{a},\ \frac{d\rho}{ds} = -\frac{r}{q^2} = -\frac{3h}{a}.$$

Thus $\quad \rho\left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\}^{-\frac{3}{4}} = \frac{1}{a}\left\{\frac{a^2}{a^2+h^2}\right\}^{\frac{3}{4}} = \frac{a^{\frac{1}{2}}}{(a^2+h^2)^{\frac{3}{4}}}.$

On transferring the equation of the hyperbola to its centre, it becomes

$$a(x^2 - y^2) + 2hxy + \frac{a}{(a^2+h^2)} = 0.$$

The lengths of the semi-axes are given by

$$\frac{1}{r^4} - \frac{(a^2+h^2)^3}{a^2} = 0.$$

Hence the result.

Additional Solution by J. C. Swaminarayan, M.A.

Question 373.

(D. D. Kapadia, M.A., B.Sc.) :—Prove that—

$$\begin{vmatrix} x-a_1, & a_1, & a_1 & \ldots & \ldots & a_1 \\ a_2, & x-a_2, & a_2, & \ldots & \ldots & a_2 \\ \ldots & \ldots & \ldots & & & \ldots \\ \ldots & \ldots & \ldots & & & \ldots \\ a_n, & a_n, & a_n, & & \ldots & x-a_n \end{vmatrix} \equiv f(x)\left\{1+\Sigma\left(\frac{a_r}{x-2\,a_r}\right)\right\},$$

where $f(x)$ stands for $(x-2a_1)\,(x-2a_2)\,(x-2a_3)\;\ldots\;\ldots\;(x-2a_n.)$

Solution by J. C. Swaminarayan, M.A., and others.

Let D denote the given determinant.

Then $D=0$, is the eliminant of ξ_1, ξ_2 &c., from the equations

$$\begin{aligned} (x-a_1)\,\xi_1+a_1\,\xi_2+a_1\,\xi_3+ \;\ldots\;\ldots\; a_1\,\xi_n&=0. \\ a_2\,\xi_1+(x-a_2)\,\xi_2+ \;\ldots\;\ldots\;\ldots\; a_2\,\xi_n&=0. \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_n\,\xi_1+a_n\,\xi_2+ \;\ldots\;\ldots\; (x-a_n)\,\xi_n&=0. \end{aligned}$$

These equations can be put in the form

$$\begin{aligned} (x-2a_1)\xi_1&=-a_1(\xi_1+\xi_2+\;.\;.\;.\;\xi_n) \\ (x-2a_2)\xi_2&=-a_2(\xi_1+\xi_2+\;.\;.\;.\;\xi_n) \\ \cdot\quad\cdot\quad\cdot\quad&\cdot\quad\cdot\quad\cdot\quad\cdot \\ (x-2a_n)\xi_n&=-a_n(\xi_1+\xi_2+\;.\;.\;.\;\xi_n). \end{aligned}$$

$$\therefore\quad \frac{a_1}{x-2a_1}+\frac{a_2}{x-2a_2}+\;.\;.\;.\;+\frac{a_n}{x-2a_n}$$

$$=-\left\{\frac{\xi_1+\xi_2+\;.\;.\;.\;+\xi_n}{\xi_1+\xi_2+\;.\;.\;.\;+\xi_n}\right\},$$

$$=-1.$$

$\therefore\; 1+\sum\frac{a_r}{x-2a_r}=0$, is also the eliminant of the same equations.

Therefore, clearing of fractions, we get

$$D\equiv\lambda f(x)\left\{1+\sum\frac{a_r}{x-2a_r},\right.$$

where λ is numerical.

Comparing the coefficients of x^n, we get $\lambda=1$.

Hence

$$D=f(x)\left\{1+\sum\frac{a_r}{x-2a_r}\right\}.$$

Question 375.

(KESHAVLAL, D. SHAH) :—Find the rational intregral factors of x^8-4x^4+16, shewing how the method adopted is suggested.

Solution (1) *by V. Ramesam, T. P. Trivedi, and N. P. Pandya*; *and* (2) *by J. C. Swaminarayan, M.A.*

(1) Since there are no terms containing x^7, x^5, x^3, and x, we may put

$$x^8-4x^4+16=(x^4+ax^3+bx^2+cx+4)(x^4-ax^3+bx^2-cx+4)$$

Equating co-efficients of x^6 and x^2, we have,

$$2b-a^2=0$$
$$8b-c^2=0.$$

$\therefore$ $c^2=4a^2$ and $c=2a$; $2ac=4a^2=8b$.

Equating co-efficients of x^4.

$$b^2-2ac+8+4=0,$$

or $$b^2-8b+12=0.$$

$\therefore$ $b=2$ or 6; the latter value does not give rational values for a and c.

$\therefore$ $b=2$ and $a=2$, $c=4$.

$\therefore$ $x^8-4x^4+16=(x^4+2x^3+2x^2+4x+4)(x^4-2x^3+2x^2-4x+4)$.

More generally,

$$x^8+2x^4(2a^4-4a^2d+d^2)+d^4$$
$$=(x^4+2ax^3+2a^2x^2+2adx+d^2)(x^4-2ax^3+2a^2x^2-2adx+d^2).$$

(2) $$x^8-4x^4+16=\frac{x^{12}+4^3}{x^4+4},$$
$$=\frac{(x^6+4x^3+8)(x^6-4x^3+8)}{(x^2+2x+2)(x^2-2x+2)},$$
$$=\frac{\{x^6-8x^3+8+12x^3\}\{x^6+8x^3+8-12x^3\}}{(x^2-2x+2)(x^2+2x+2)},$$
$$=(x^4+2x^3+2x^2+4x+4)(x^4-2x^3+2x^2-4x+4),$$

Question 376.

(ZERO) :—In the limacon $r=a+b\cos\theta$, if r_1, r_2, r_3, r_4 denote the radii to the points of contact of parallel tangents, prove that

(i) $r_1+r_2+r_3+r_4=3a$.

(ii) $\Sigma(r_1r_2)$ is invariable.

Solution by T. P. Trivedi, and others.

We have $$r=a+b\cos\theta \quad \ldots \quad \ldots \quad (1)$$

$$\therefore \quad \frac{rd\theta}{dr}=\tan\phi=-\frac{a+b\cos\theta}{b\sin\theta}.$$

If ψ be the inclination of the tangent to the initial line

$$\psi=\theta-\tan^{-1}\frac{a+b\cos\theta}{b\sin\theta}.$$

$$\tan\psi=\frac{b\sin^2\theta-a\cos\theta-b\cos^2\theta}{\sin\theta(a+2b\cos\theta)}=k,\text{ say } \dots \quad (2)$$

Eliminating θ between (1) and (2), we get

$$(2r^2-3ar+a^2-b^2)^2-k^2\{b^2-(r-a)^2\}(2r-a)^2=0,$$

as the equation to determine the radii vectors to the points of contact.

If r_1, r_2, r_3, r_4 be the roots

(i) $$\Sigma r_1=\frac{12a(1+k^2)}{4(1+k^2)}=3a,$$

(ii) $$\Sigma(r_1r_2)=\frac{(13a^2-4b^2)(1+k^2)}{4(1+k^2)}=\frac{13a^2-4b^2}{4}=\text{constant}.$$

Question 377.

(J. C. SWAMINARAYAN, M.A.):—Shew that the value of the determinant—

$$\begin{vmatrix} a, & h, & g, & l, & l' \\ h, & b, & f, & m, & m' \\ g, & f, & c, & n, & n' \\ l, & m, & n, & o, & o \\ l', & m', & n', & o, & o \end{vmatrix}$$

can be expressed in either of the forms

(1) $\Sigma a(mn'-m'n)^2+2\Sigma f(nl'-n'l)(lm'-l'm)$,

(2) $\frac{1}{\Delta}[(\Sigma Al^2+2\Sigma Fmn)(\Sigma Al'^2+2\Sigma Fm'n')-$

$\{\Sigma A\, ll'+\Sigma F(mn'+m'n)\}^2]$,

where corresponding capital letters denote the minors, and Δ denotes the value, of the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Solution by T. P. Trivedi, M.A., LL.B. and N. Sanjiva Rao.

Expanding the determinant in term of the minors formed by taking two columns from the last two rows, we have the determinant

$$=(lm'-l'm)\begin{vmatrix} g, & l, & l' \\ f, & m, & m' \\ c, & n, & n' \end{vmatrix} -(ln'-l'n)\begin{vmatrix} h, & l, & l' \\ b, & m, & m' \\ f, & n, & n' \end{vmatrix}$$

$$+(mn'-m'n)\begin{vmatrix} a, & l, & l' \\ h, & m, & m' \\ g, & n, & n' \end{vmatrix}.$$

Simplifying and expanding in terms of the co-efficients of $a,b,c,f,g,h,$ we get the first result.

Again, in the expression

$$[(\Sigma Al^2+\Sigma Fmn)(\Sigma Al'^2+2\Sigma Fm'n')-\{\Sigma All'+\Sigma F(mn'+m'n)\}^2],$$

the co-efficient of BC is $m^2n'^2+m'^2n^2-2mm'nn'=(mn'-m'n)^2$; the co-efficient of F^2 is $4mn\,m'n'-(mn'+m'n)^2=-(mn'-m'n)^2$.

Thus the expression is seen to be equal to

$$\Sigma(BC-F^2)(mn'-m'n)^2+2\Sigma(H^2-AB)(nl'-n'l)(lm'-l'm)$$

But $BC-F^2$ = minor of A in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = a\Delta, \text{ as is known.}$$

Making similar substitutions, the expression $=\Delta$ (the value of the determinant).

Hence the second result.

Note by K. J. Sanjana, M. A.

The vanishing of the determinant of the question gives the condition that the lines $\Sigma lx=0$, $\Sigma l'x=0$, should interest on the conic $\Sigma ax^2+2\Sigma fyz=0$, and is stated by Casey, § 261. It may be directly obtained as follows :—

$\Sigma lx+\lambda\Sigma l'x\equiv\Sigma(l+\lambda l')x=0$, is some line through the intersection of the given lines. Its pole with regard to the given conic is determined by the three equations

$$ax+hy+gz-b-\lambda l'=0,$$
$$hx+by+fz-m-\lambda m'=0,$$
$$gx+fy+cz-n-\lambda n'=0.$$

Let this pole be at the intersection of the two lines; then it is on the polar itself, and consequently on the conic, as the polar is now a tangent. But the pole thus satisfies the two additional conditions

$$lx+my+nz=0,$$
$$l'x+m'y+n'z=0,$$

Eliminating x, y, z, -1, $-\lambda$ from the five equations, we get the condition in the required form.

QUESTIONS FOR SOLUTION.

400. (D. D. KAPADIA, M.A., B.Sc.) :—Prove that if $|x|<1$,

$$\frac{x}{(1+x)^2}+\frac{x^2}{(1+x^2)^2}+\frac{x^3}{(1+x^3)^2}+\cdots\cdots\cdots$$

$$=\frac{x}{1-x}-\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}-\cdots\cdots=\sum_1^\infty \psi(\mu)x^\mu,$$

where $\psi(\mu)$ denotes the excess of the odd over the even divisors of μ (1 and μ being reckoned as divisors).

401. (S. P. SINGARAVELU MUDELIAR, B.A.):—A straight line is drawn in the plane of a parabola through the foot of the directrix. If the plane of the parabola be vertical and its axis horizontal, shew that the time of quickest descent from the straight line to the parabola is $\sqrt{2l\cos 2\theta / g}(2\cos\theta+\sin 2\theta)$, where θ is the inclination of the straight line to the vertical and l the latus rectum of the parabola.

402. (J. C. SWAMINARAYAN, M. A.) :—Prove that

$$\int_0^1\left(\frac{\sqrt{1-z^2}}{z}\right)(\tanh^{-1}z)dz=\frac{\pi^2}{4}-\frac{\pi}{2}$$

403. (S. NARAYAN, B.A., L.T.) :—Shew that the square of the semi-axis of the maximum inscribed ellipse of a triangle is $\frac{1}{9}\Delta\cot\lambda$, where Δ is the area of the triangle, and λ the semi-steiner angle (the smaller corresponding to the major axis, and the greater to the minor axis).

404. (A. C. L. WILKINSON, M.A., F.R. A.S.) :—In a tetrahedron, for which the products of the pairs of opposite edges are equal, prove that any plane parallel to the tangent plane to the circumscribing sphere at one of the vertices, cuts the three edges through that vertex in an equailateral triangle.

405. (M. V. ARUNACHALA SASTRY, M.A.) :—Construct a trapezium ABCD, which can have a circle inscribed in it, given one of the parallel sides AB in position and magnitude, the length of the parallel side, and a line L on which the oblique sides intersect.

406. (K. J. SANJANA, M.A.) :—If m_1, m_2, m_3 the medians and s_1, s_2, s_3 the symmedians of a triangle ABC, prove that

(i) $\Sigma[a(b^2+c^2).\ s_1m_1]=\frac{3}{2}abc(a^2+b^2+c^2)$;

(ii) $a(b^2+c^2).\ \frac{s_1}{m_1}+b(c^2+a^2).\ \frac{s_2}{m_2}+c(a_2+b_2).\ \frac{s_3}{m_3}=24\Delta R.$

407. (D. D. KAPADIA, M.A., B.Sc.) :—The polars of the point (x,y,z) with respects to the three conics $S_1=0$, $S_2=0$, $S_3=7$, are concurrent ; prove that the point (x, y, z) as well as the point of concurrence lie on the Jacobian

$$\sum\pm\frac{\partial S_1}{\partial x}\cdot\frac{\partial S_2}{\partial y}\cdot\frac{\partial S_3}{\partial z}=0.$$

408. (J. C. SWAMINARAYAN, M.A.):—A regular polygon $A_1A_2A_3\ldots A_m$ of m sides and a regular polygon $B_1B_2B_3\ldots B_n$ of n sides are inscribed in a circle of radius r. Shew that the continued product of all the chords AB is numerically equal to

$$2r^{m+n}\sin(m+n)\theta$$

where 2θ, is the angle subtended by one of these chords at the centre of the circle. (Extension of Q. 389 of Mr. D. D. Kapadia, M.A., B.Sc.).

409. (S. P. SINGARAVELU MOODELIAR, B.A.) :—Q is any point from which four normals are drawn to the ellipse $ax^2+by^2=1$, whose centre is C. If N_1, N_2, N_3, N_4 are the projections upon CQ of P_1, P_2, P_3, P_4, the feet of the normals, shew that

$$\Sigma CP_1^2-CQ\cdot\Sigma CN_1=2(1/a+1/b).$$

410. (P. A. SUBRAMANIA IYER, B.A., L.T.) :—Solve completely

$$\left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2\right\}=a\left[1+\left(\frac{dy}{dx}\right)^2\right]^3.$$

411. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—Show that the evolute of the curve whose intrinsic equation is $\frac{s^2}{a^2}-\frac{\rho^2}{b^2}=1$, is $\frac{\rho^2a^2}{b^4}-\frac{s^2}{b^2}=1$.

Determine the forms of both of these curves. Show that for both the curves a point O exists such that the projection of the radius vector from O on the tangent and normal at any point are proportional to s and ρ respectively. Examine more closely the meaning of s in the first of these equations.

412. (MARTYN M. THOMAS) :—Apply the theory of moving axes to solve the following: "At every point of a plane curve a line is drawn making a given angle α with the normal; let the envelope of the line be termed the α—evolute of the curve. Prove that the α—evolute of the β—evolute of any curve is the β—evolute of the α—evolute of the same." [M.A. Degree Examination, 1911, Madras.]

413. (M. KANNAN, B.A., L.T.) :—Shew that the Apollonian Circle through the vertex A of a triangle ABC cuts the Nine-point circle and the Polar circle of the triangle at $\cos^{-1}(\cos A \sin \overline{C-B})$ and $\cos^{-1}\left(\frac{R}{\rho}\cos A \sin\overline{C-B}\right)$, respectively, ρ being the radius of the Polar Circle.

414. (R. TATA, M.A.) :—If DEF is the pedal triangle of ABC, and D′E′F′ that of its symmedian point, shew that

$$a.\ DD'+b.\ EE'+c.\ FF'=0.$$

415. (A. A. KRISHNASWAMI AIYANGAR) :—If $s_r=1^r+2^r+\dots n^r$, and $c_p={}^rC_p$, shew that

$$1+c_0s_0+c_1s_1+c_2s_2+\dots c_{r-1}\ s_{r-1}=(1+n)^r.$$

416. (M. T. NARANIENGAR) :—If l, l' ; m, m' ; n, n' are the distances of the foci of an inconic from the angular points, shew that the areal co-ordinates of the centre of the conic are

$$\tfrac{1}{4}(m\ m' \sin B+n\ n' \sin C), \text{ \&c.}$$

417. (V. RAMASWAMI AIYAR, M.A.) :—If parabolas are escribed to a triangle, shew that any system of corresponding lines connected with them (considered as similar figures) envelope a three-cusped hypocycloid.